Mathematics 339
April 14, 2009
Professor Cownden
Final Exam

Please read the following points carefully before proceeding!

1. Please note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.
2. You are not allowed any aids other than those included on the last page of this exam and a casio 991 caluclator or equivalent (gold sticker).
3. Please do not write in red.
4. Print your student number at the top of this page. Do not provide your name.
5. You have 3 hours to complete the exam.
6. Please read all questions carefully before answering them.
7. Answer all questions on the exam paper (both sides); any work not on the exam paper will not be graded. I have tried to ensure that you have enough space on the exam paper to answer each question, but do not feel as if your answer needs to fill out the available space in order to be correct!
8. Marks per question are shown in square brackets after the problem number. The total is 70.
9. Check that your question paper has ?? pages.

Classifying Equilibria with a Jacobian

Suppose you have a system of differential equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f(p,q)$$
$$\frac{\mathrm{d}q}{\mathrm{d}t} = g(p,q)$$

Then the Jacobian of this system is defined to be

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then let $D = (ad - bc)$ and $T = (a + d)$. If the Jacobian is evaluated at an equilibrium point of the system then using D and T we can classify the type of equilibirum as follows.

1. If $D = 0$ can't say anything about the equilibrium from the Jacobian.
2. If $D < 0$ the equilibrium is a saddle point, and hence unstable. No need to compute T.
3. If $D > 0$ and $T < 0$ the equilibrium is asymptotically stable.
4. If $D > 0$ and $T > 0$ the equilibrium is asymptotically unstable.
5. If $D > 0$ and $T = 0$ the equilibrium point is centre, which is stable but not asymptotically stable. (i.e. not the kind of stable that corresponds to an ESS)
6. If $D > 0$ and $T^2 - 4D < 0$ the equilibrium point is a focus
7. If $D > 0$ and $T^2 - 4D \geq 0$ the equilibrium is a node.

Maximizing a function of two variables

Let $f(p,q)$ be a function of two variables. It's critical points are those which satisify both of the follwoing equations.

$$\frac{\partial f}{\partial p} = 0$$
$$\frac{\partial f}{\partial q} = 0$$

To check if a critical point is a local maximum, local minimum, or a saddle point you need to look at the matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial p^2} & \frac{\partial^2 f}{\partial p \partial q} \\ \frac{\partial^2 f}{\partial q \partial p} & \frac{\partial^2 f}{\partial q^2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then let $D = ad - bc$ if $D < 0$ the critical point is a saddle point, if $D > 0$ and $\frac{\partial^2 f}{\partial p^2} < 0$ then the critical point is a maximum, if $D > 0$ and $\frac{\partial^2 f}{\partial p^2} > 0$ then the critical point is a minimum.

Binomial distribution

Suppose a coin has a p probability of coming up heads and it is flipped n times. Then the probability of exactly i flips coming up heads is given by $\binom{n}{i}p^i(1-p)^i$. If your calculator doesn't do this for you automatically $\binom{n}{i} = \frac{n!}{(n-i)!\cdot i!}$. This might be helpful in problem one.