Solutions to proto-final

Daniel

April 13, 2009

1 To Sue or settle

1. If you draw out the game tree you can quickly see that my expected value for going to court is \$75 million and McDonalds expected value from going to court is \$-78 million. So assuming we are both trying to maximize expected payoff they will never offer a settlement of more than \$78 million and I will never accept an offer of below \$75 million. Hence the sub-game perfect Nash equilibrium in this game is for McDonalds to always offer me \$ $75 + \epsilon$ million or if only integer amounts are if I am risk averse then they can offer me \$ 75 million and I will take it, or if I am risk loving they need to offer me \$ 76 million, and for me to accept any offers that are greater than \$ 75 million (or equal if I am risk averse). So assuming we are Nash equilibrium playing types we do reach a settlement, and it is that they give me 75 or 76 million.

2. Now my expected value for going to court is 68 million (since now there is a chance I'll have to cough up some money) and McDonald's expected value is -64 million, so there is no settlement they can offer me that will be better for both of us.

3. Even though the new rule provided me with a dissinsentive to go to court, it lessened McDonalds disensentive more than it increased mine, so in fact it actually disouraged settlement. And it is not my disensentive alone that matters in terms of reaching a settlement but both mine and McDonalds, and the difference between them. Note that if I had a less belief in my chance to win than McDonald's did this would work.

2 Who Called the Cops

1. If one person calls they get $u - c$ if they don't call they get 0, so if there is only one denzien then the Nash equilibrium is for that person to always call.

2. If there are only two denziens then we can write things out as a matrix game like so:

	C	N
C	$(u-c),(u-c)$	$(u-c),u$
N	$u,(u-c)$	(0,0)

Clearly there are two pure strategy Nash equilibria, both of the form one player calls and the other player does not call. There is also a mixed strategy Nash equilibrium, of the form (σ^*, σ^*) where $\sigma^* = p^*\text{call}, (1 - p^*)$not call Using the FTMSNE (fundamental theorem of mixed strategy Nash equilibrium) we have that $\pi(C, \sigma^*) = \pi(N, \sigma^*)$ that is the payoff against the MSNE strategy is the same regardless of whether you play call or not call. Expanding this we get that: $u - c = p^* u$ so that $p^* = 1 - \frac{c}{u}$ which gives us the MSNE.$\sigma^* = 1 - \frac{c}{u}$call, $\frac{c}{u}$not call.

3. If there are an arbitrary number of Denizens, say n, then the payoff to a denizen playing call will always be $\pi(C, \sigma) = u - c$ where σ is an arbitrary profile of what all the other denizens are doing. Now $\pi(N, \sigma) = u$ if at least one other person calls, but $\pi(N, \sigma) = 0$ if no one else calls so again using the FTMSNE we have that

$$\begin{aligned}\pi(C, \sigma) &= \pi(N, \sigma) \\ u - c &= 0(1 - p^*)^{n-1} + u(1 - (1 - p^*)^{n-1}) \\ \frac{c}{u} &= (1 - p^*)^{n-1} \\ p^* &= 1 - \sqrt[n-1]{\frac{c}{u}}\end{aligned}$$

Where did this come from, well no matter what everyone else does if a denizen calls for help then regardless of what anyone else does, they will get $u - c$. If a player does not call and no one else calls which happens with probability $(1 - p^*)^{n-1}$ then that player gets zero, if at least one person calls which happens with probability $(1 - (1 - p^*)^{n-1})$ then the non-caller gets u. Now there are also n pure strategy Nash equilibria each of the form, one person calls and everyone else does not.

4. Supposing that people play the mixed strategy Nash equilibrium, then they will be less likely to call as the number of people in the neighbourhood increases, however the probability that anyone calls is $1 - (\sqrt[n-1]{\frac{c}{u}})^n = \frac{c}{u}^{\frac{n}{n-1}}$ converges to a constant $\frac{c}{u}$. So if someone asked my what size neighboorhood I would like to be mugged in I would have to pick a Neighbourhood of size one, after that all big neighbourhoods are about the same.

3 Matrix Game Acrobatics

1. If $x > 3$ and $y > 2$ then (s_1, t_1) is a strict Nash equilibrium.

2. If $x = 3$ or $y = 2$ or $x > 3$ and $y > 2$ then (s_1, t_1) is a weak Nash equilibrium.

3. Yet again we use the FTMSNE. From it we get that

$$\begin{aligned}\pi(s_1, \tau^*) &= \pi(s_2, \tau^*)\\ xq^* + 2(1-q^*) &= 3q^* + 3(1-q^*)\\ x\frac{1}{3} + \frac{2}{3} &= 3\\ x + 2 &= 7\\ x &= 5\end{aligned}$$

and also that

$$\begin{aligned}\pi(t_1, \sigma^*) &= \pi(t_2, \sigma^*)\\ yp^* + 1(1-p^*) &= 2p^* + 4(1-p^*)\\ y\frac{3}{4} + \frac{1}{4} &= 2(\frac{3}{4}) + 4\frac{1}{4}\\ y3 + 1 &= 6 + 4\\ y &= 3\end{aligned}$$

4. Plugging things in we see that the expected value for the s player is 3 and the payoff for the t player is $\frac{10}{4}$

4 Continuous Strategies

1. To find player one's best response, we need to figure out what player one (the x player) should do given what player two (the y player) has done. $\pi_1(x, y) = -x^2 + 6x + y$ so we need to find the x which maximizes this function for a given y. So we calculate $\frac{\partial x}{\partial \pi_1} = -2x + 6$ setting this to zero and solving we see that for all y the payoff function has a critical point at $x = 3$ and that this point is indeed a maximum since $\frac{\partial^2 x}{\partial {\pi_1}^2} = -2 < 0$. The value of the function at this point is then $\pi_1(3, y) = 9 + y$. Since the second derivative of the payoff function is always negative we don't need to worry about the boundary points being maxima. To find player two's best response we use the same technique. So we calculate $\frac{\partial y}{\partial \pi_2} = \frac{-2y}{x} + 1$ setting this to zero and solving we see that for all x within the allowable range of x the payoff function has a critical point at $y = \frac{x}{2}$, and that this point is indeed a maximum since $\frac{\partial^2 y}{\partial {\pi_2}^2} = \frac{-2}{x} < 0$.for all allowable x. Again we don't need to worry about the boundary points as they will be minimums and not maximums since the second derivative of the payoff function is negative there. So $B_x(y) = 3$ and $B_y(x) = \frac{x}{2}$.

2. so clearly the intersection of these two best response functions is the point $(x, y) = (3, \frac{3}{2}$ The payoff to each of these players at this point is $\pi_1(3, 1.5) = 10.5$ and $\pi_2(3, 1.5) = 0.375$.

5 Knife edged saddle point

1. First let $\sigma = pC, (1-p)G$ be a mixed strategy for deer and let $\tau = qT, (1-q)B$ using the FTMSNE we have that

$$\begin{aligned} \pi(C, \tau^*) &= \pi(G, \tau^*) \\ 3q^* &= 2q^* + 1(1-q^*) \\ q^* &= \frac{1}{2} \end{aligned}$$

and we have that

$$\begin{aligned} \pi(T, \sigma^*) &= \pi(B, \sigma^*) \\ 2p^* + 1(1-p^*) &= 3(1-p^*) \\ p^* &= \frac{1}{2} \end{aligned}$$

So the MSNE is (σ^*, τ^*) where $\sigma^* = \frac{3}{4}C, \frac{1}{4}G$ and $\tau^* = \frac{1}{2}T, \frac{1}{2}B$.

2. The replicator equations are

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} &= p(\pi_C - \bar{\pi}) \\ &= p(3q - (p3q + (1-p)(2q + (1-q)) \\ &= p(3q - (3pq + (q + 1 - pq - p)) \\ &= p(2q - 2pq - 1 + p) \\ &= p(2q(1-p) - (1-p)) \\ &= p(1-p)(2q-1) \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= p(\pi_T - \bar{\pi}) \\ &= q(1 + p - (q(1+p) + (1-q)(3(1-p)) \\ &= q(1 + p - (q + qp + 3 - 3q - 3p + 3pq)) \\ &= q(-2 + 4p + 2q - 4pq) \\ &= q(-2(1-q) + 4p(1-q)) \\ &= q(1-q)(4p-2) \end{aligned}$$

3. Setting the replicator equations to zero we see that the p isoclines are $p = 0, p = 1, \text{and} q = \frac{1}{2}$ and the q isoclines are $q = 0, q = 1, \text{and} p = \frac{1}{2}$

4. The equilibria then are $(0,0), (0,1), (1,0), (1,1) \text{and} (\frac{1}{2}, \frac{1}{2})$

5. the Jacobian is

$$\begin{pmatrix} (1-2p)(2q-1) & 2p(1-p) \\ 4q(1-q) & (1-2q)(4p-2) \end{pmatrix}$$

evaluated at (0,0) we get

$$\begin{pmatrix} (1)(-1) & 0 \\ 0 & (1)(-2) \end{pmatrix}$$

$D = 2$ and $T = -4$ and $T^2 - 4D = 16 - 8 = 8 > 0$ so (0,0) is a stable node.

evaluated at (0,1) we get

$$\begin{pmatrix} (1)(1) & 0 \\ 0 & (-1)(-2) \end{pmatrix}$$

$D = 2$ and $T = 3$ and $T^2 - 4D = 9 - 8 = 1 > 0$ so (0,1) is an unstable node

evaluated at (1,0) we get

$$\begin{pmatrix} (-1)(-1) & 0 \\ 0 & (1)(2) \end{pmatrix}$$

$D = 2$ and $T = 3$ and $T^2 - 4d = 9 - 8 = 1 > 0$ so (1,0) is also an unstable node.

evaluated at (1,1) we get

$$\begin{pmatrix} (-1)(1) & 0 \\ 0 & (-1)(2) \end{pmatrix}$$

$D = 2$, and $T = -3$, and $T^2 - 4D = 9 - 8 = 1$ so (1,1) is an asymptotically stable node.

evaluated at $(\frac{3}{4}, \frac{1}{2})$

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$D = -\frac{1}{2} < 0$ so this is a saddle point.

6. I don't have access to a scanner and the train is too bumpy to even try to use a paint program so you'll have to wait for a picture.

7. Here's the story there is some critical boundary that runs through the space and passes through the MSNE point. If the population starts on one side of this boundary then there are enough Tasty ferns around and enough Considerate munchers, that they can synergistically help eachother out and win out against the Bitter ferns and Gorging deer, and the system will converge to the point (1,1), where the whole fern population is tasty, and the whole deer population is considerate. If the population starts on the other side of this boundary then there are not enough Considerate munchers to make being Tasty worthwhile and there are not enough Tasty ferns to make being Considerate worthwhile for a deer, so the system will converge to the point (0,0) where every deer is a Gorger and every fern is Bitter.

6 working backwards

so there isn't much point in giving you a solution to this, but I will tell you how to prove an MSNE is an ESS. Suppose you had to deal with a 3 by 3 symetric matrix game

by the FTMSNE property you have that $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ where $\sigma^* = p^* s_1, q^* s_2, (1 - p^* - q^*) s_3$ is the MSNE and $\sigma = p^* s_1, q^* s_2, (1 - p^* - q^*) s_3$ is an arbitrary mixed strategy. So we know we have to use the other ESS condition (since MSNE are never strict Nash equilibria). So in order for the MSNE to be an ESS the following inequality must be satisfied

$$\pi(\sigma^*, \sigma) \geq \pi(\sigma, \sigma)$$
$$\iff p^* \pi(s_1, \sigma) + (q^*) \pi(s_2, \sigma) + (1 - p^* - q^*) \pi(s_3, \sigma) \geq p \pi(s_1, \sigma) + (1 - p) \pi(s_2, \sigma) + (1 - p - q) \pi(s_3, \sigma)$$

with equality only when $\sigma = \sigma^*$. you then solve this for an expression of p and q. and then use the technique on the cheat sheet to prove that inequality is true.