Mathematics 339
April 2010
Professor Cownden
Practice Exam

Please read the following points carefully before proceeding!

1. Please note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.
2. You are not allowed any aids other than those included on the last pages of this exam and a Casio 991 calculator or equivalent (gold sticker).
3. Please do not write in red.
4. Print your student number at the top of this page. Do not provide your name.
5. You have 3 hours to complete the exam.
6. Please read all questions carefully before answering them.
7. Answer all questions on the exam paper (both sides); any work not on the exam paper will not be graded. I have tried to ensure that you have enough space on the exam paper to answer each question, but do not feel as if your answer needs to fill out the available space in order to be correct!
8. Marks per question are shown in square brackets after the problem number. The total is 75.
9. Check that your question paper has 9 pages.

1 The Final Game [15]

Everyone writing the MATH 339 exam has been given this same question. For the purposes of this question assume that there are N students in total writing the exam. There are 80 students enrolled but who knows if they've all shown up.

The game is as follows, each exam writer must select a positive integer (1,2,3...). To win a player must choose the lowest integer n with the property that fewer than n other players have selected the integer n. For example if four people are playing, and one player chooses 1, two players choose 2 and 1 player chooses 3, the player who choses 1 is the winner and everyone else loses. If on the other hand two players chose 1 and 2 players choose 2 the players who chose 2 would be winners.

1. Find two different Nash Equilibrium for this game, for which all exam writers win.[5]
2. Suppose that $N = 3$. Find all of the mixed strategy Nash equilibria. (Hint use dominated strategies to see which integers should be played with probability zero.)[5]
3. Suppose that every player uses a mixed strategy where they play integer n with probability p_n. (Note that $\sum_{n=1}^{N}(p_n) = 1$) Write out an expression for the probability of winning in an N player game as a function of the p_n's.[5]

answer

1. If everyone plays the integer N then everyone will win. However, while there is no incentive for a single player to defect to playing any smaller integer, there is also no incentive for them not to switch (except that they screw everyone else over, but empathy is not apart of the payoff function). So this is a weak Nash equilibrium. Similarly if everyone plays the integer $N+1$ everyone will win.
2. Let p_n be the probability of playing integer n. If only 3 people are in the game than choosing 3 is gaurenteed to beat any choice of a higher integer. Given the p_n's the probability of winning when playing 1 is the probability that both players choose a higher integer which is $(1-p_1)^2$. The probability of winning when playing 2 is the probability that the other two people choose one plus the probability that no one chooses 1 and that no more than other 1 person chooses 2, which is $p_1^2 + p_2 p_3 \binom{2}{3} + p_3^2$. The probability of winning when a player chooses 3 is the probability that everyone else chooses 3, which is p_3^2. Comparing the probabilities of winning when choosing the integer 2 versus choosing the integer 3 we can see that choosing 2 weakly dominates choosing 3, since for any non zero value of p_2 the proability of winning when choosing 2 is greater than when choosing 3, and that they are equal when p_2 is zeron. From this we can deduce that only 1 and 2 will be chosen with positive probability. At the MSNE the probability of winning by choosing 1 must be equal to the probability of winning by choosing 2 and the probabilities of choosing 1

and 2 must sum to 1. So calulate the MSNE strategy, we simply solve this system of equations.

$$(1 - p_1)^2 = p_1^2 \tag{1}$$
$$p_1 + p_2 = 1 \tag{2}$$

Solving we get that $p_1 = p_2 = \frac{1}{2}$. So when every player chooses 1 with probability $\frac{1}{2}$ and chooses 2 with probability $\frac{1}{2}$ then no player has incentive to change the probability with which they choose integer 1 or integer 2. There are no other mixed strategies.

3. In general the probability of winning in a N player game is that either every integer below your choice has been overselected, or not selected, and that your integer has not been overselected. The probability that a given integer is overselected is $\sum_{i=n+2}^{N-1} p_n^i (1 - p_n)^{N-1-i} \binom{N-1}{i}$. This problem is way harder than anything on the exam, a more reasonable question would have been what is the probability for $N = 3$. I'll answer that. The probability of winning when choosing 1 is $(1 - p_1)^2$. The probability of winning when choosing 2 is $(1 - p_1 - p_2)^2 + p_1^2 + p_2(1 - p_1 - p_2)\binom{2}{1}$. The probability of winning when choosing 3 is $(1-p_1-p_2)^2+p_1^2)$ and in general the probability of winning when choosing n is $(1 - \sum_{i=1}^{n} p_i)^2 + p_1^2$.

2 Two-Player Polynomials [20]

Player one must choose a real number, x, between -2 and 2 inclusive. Player two must choose a real number, y, between -3 and 3 inclusive. The payoff to player one is given by

$$\Pi_1(x, y) = x^3y - xy^3 \tag{3}$$

and the payoff to player two is given by

$$\Pi_2(x, y) = x^2y - xy^2 \tag{4}$$

1. Find player one's best response to a given value of y.Hint: This function will be piecewise continuous conditional on the value of y.[5]
2. Find player two's best response to a given value of x.[5]
3. Find all the pairs of values of x and y where both players are playing the best response to eachother, aka the intersection of the best response functions, aka the Nash equilibria.[8]
4. What is the best equilibrium for player one, and what is the best equilibrium for player two?[2]

answer

1. For a given value of y what x value maximizes $\Pi_1(x, y)$. First we take the derinvative with respect to x, set it equal to zero and solve for the critical x values.

$$\frac{\partial x}{\partial \Pi_1} = 3x^2y - y^3 = 0$$
$$x^2 = \frac{y^2}{3}$$
$$x = \pm\frac{y}{\sqrt{3}}$$

   Looking at the second derivative to see which critical points and boundary points are max and min.

$$\left.\frac{\partial^2 x}{\partial {\Pi_1}^2}\right|_{x=\frac{y}{\sqrt{3}}} = \frac{y^2}{\sqrt{3}}$$
$$\left.\frac{\partial^2 x}{\partial {\Pi_1}^2}\right|_{x=\frac{-y}{\sqrt{3}}} = \frac{-y^2}{\sqrt{3}}$$

   So the maxima of the payoff function occur at the point $x = \frac{-y}{\sqrt{3}}$ and the boundary point $x = 2$. Which maxima is greater for which values of y? We need to compare and see. $\Pi_1(2, y) = 8y - 2y^3$ and $\Pi_1(\frac{-y}{\sqrt{3}}, y) = \frac{y^4}{\sqrt{3}}\frac{1}{3}$ These two things are equal when $y = 0$, $y \approx 1.73$, $y \approx -3.54$ and $y \approx -3.339$ (I found those last three using a cubic root finder... you won't be expected to do that on your exam). y can only be between -3 and 3 and using our knowledge qubics and quadratics we can deduce that for $-3 < y < 0$ $x = \frac{-y}{\sqrt{3}}$ is the best response and for $0 < y < 1.73$ $x = 2$ is the best response, and for $1.73 < y < 3$ $x = \frac{-y}{\sqrt{3}}$ is the best response.

2. for a given value of x what is the y value that maximizes $\Pi_2(x, y)$. First we take the derinvative with respect to y, set it equal to zero and solve for the critical y values.

$$\frac{\partial y}{\partial \Pi_2} = x^2 - 2xy = 0$$
$$y = \frac{x}{2}$$

Looking at the second derivative to see which critical points and boundary points are max and min.

$$\frac{\partial^2 y}{\partial {\Pi_2}^2} = -2x$$

So the critical point is a maxima when x is positive and the boundary points are maxima when x is negative. Clearly the right boundary point is better so the best response of y to x is to play 3 if $x < 0$ and to play $\frac{x}{2}$ if $0 < x$.

3. I will draw a picture of the best response functions in the help sessions. They intersect (weakly) at $x = 0$ $y = 0$ so there is a weak nash eq there, and thats the only one. The problem on the final is more interesting, and a little less hard.

4. there's only the one equilibrium and so this last question makes no sense.

3 ESS and Replicator Equations [40]

Consider the following symmetric two-player matrix game.

	A	B	C
A	(1,1)	(3,4)	(0,2)
B	(4,3)	(0,0)	(0,2)
C	(2,0)	(2,0)	(1,1)

1. Find all the pure strategy Nash equilibria for this game.[2]
2. Under what conditions is a strategy an Evolutionarily Stable Strategy (ESS).[2]
3. Are any of the pure strategies being played at the Nash equilibria you found, Evoltionarily Stable Strategies. Hint: Evolutionary stability is a property of strategies. An equilibrium is only evolutionarily stable if the (mix of) strategies being played there (is) are evolutionarily stable.[4]
4. Let p be the proportion of the population playing A and let q be the proportion of the population playing B, note that the proportion of the population playing C will be $1 - p - q$. Find a mixed strategy Nash equilibria for this game.[2]
5. Is this MSNE an ESS, show your work.[6]
6. Write out the replicator equations for this game. You should have two replicator equations one for the dynamics of p and one for the dynamics of q.[4]
7. Find the equilibria of the system. These are the points where both $\frac{\mathrm{d}p}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}q}{\mathrm{d}t} = 0$. Hint: The mixed strategy is likely an equilibrium point, as are the points where only one strategy type is present in the population.[6]
8. Calculate the Jacobian of the system and use it to determine the stability and type of each of the equilibria of the system.[6]
9. Draw a well **labled** picture including labled axis, the easy to draw p and q isoclines, every equilibrium point, and your best guess at what the trajectories look like.[6]
10. Suppose you are in a population that happens to play A with probability $\frac{1}{3}$ B with probability $\frac{1}{3}$, and C with probabilty $\frac{1}{3}$. What is the best possible strategy that you could adopt for the moment?[2]

(Hint: Since $1 - p - q$ is a proportion we have that $0 \leq 1 - p - q \leq 1$. What constraints does this place on p and q?, check that your solution is reasonable in light of these constraints.)

answer

1. (A,B), (B,A) and (C,C) are all pure strategy Nash eq.

2. A strategy, s^* is an ESS if and only if for all possible other strategies s: $\Pi(s, s^*) \leq \Pi(s^*, s^*)$ but if $\Pi(s, s^*) = \Pi(s^*, s^*)$ then either $s = s^*$ or $\Pi(s, s) \leq \Pi(s^*, s)$. Where $\Pi(s, s^*)$ is the payoff of a player playing s against a player playing the MSNE strategy s^* in a two player symmetric game.

3. C is an ESS since $\Pi(A, C) = \Pi(B, C) = 0 < 1 = \Pi(C, C)$. A in not an ESS since $\Pi(A, A) = 1 < 4 = \Pi(B, A)$. B is not an ESS since $\Pi(B, B) = 0 < 3 = \Pi(A, B)$.

4. $\Pi_A = p + 3q$, $\Pi_B = 4p$, and $\Pi_C = 1 + p + q$. If all of these are equal then there is no incentive to play any of these strategies more than any of the others. $\Pi_C = \Pi_A \Rightarrow q = \frac{1}{2}$ and subbing that into $\Pi_B = \Pi_A$ we get $p = \frac{1}{2}$. So when A is played with probability $\frac{1}{2}$ and B is played with probability $\frac{1}{2}$ and C is played with probability 0. There is no incentive for anyone to change their strategy so we have a Nash equlibrium.

5. Is $p^* = q^* = \frac{1}{2}$ and ESS? By a property of mixed strategegy Nash eq. we know that $\Pi((p, q), (\frac{1}{2}, \frac{1}{2})) = \Pi((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})) =$ constant. Therefore if $\Pi((p, q), (p, q)) < \Pi((\frac{1}{2}, \frac{1}{2}), (p, q))$ for all admissible values of p and q then the MSNE is an ESS.

$$\begin{aligned} \Pi((p, q), (p, q)) &< \Pi((\frac{1}{2}, \frac{1}{2}), (p, q)) \\ p\Pi_A + q\Pi_B + (1 - p - q)\Pi_C &< \frac{1}{2}\Pi_A + \frac{1}{2}\Pi_B \\ 5pq + 1 - q^2 &< \frac{5p + 3q}{2} \end{aligned}$$

This inequality is clearly violated when $p = q = 0$ so there are admissible p and q such that a rare mutant could invade a population playing the MSNE strategy, hence the MSNE strategy is not an ESS.

6. The replicator equations are:

$$\begin{aligned} \frac{dp}{dt} = f(p, q) &= p(\Pi_A - \bar{\Pi}) \\ &= p(p + 3q - 5pq - 1 + q^2) \\ \frac{dq}{dt} = g(p, q) &= q(\Pi_B - \bar{\Pi}) \\ &= q(4p - 5pq - 1 + q^2) \end{aligned}$$

7. Check that $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ are equilibria. Clearly $f(0, 0) = 0$ and $g(0, 0) = 0$ so $(0, 0)$ is an eq. $f(0, 1) = 0$ and $g(0, 1) = 1(0-0-1+1) = 0$ so $(0, 1)$ is an eq. $f(1, 0) = 1(1+0-0-1+0) = 0$ and $g(1, 0) = 0$ so $(1, 0)$ is an eq. $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(\frac{1}{2}+\frac{3}{2}-\frac{5}{4}-1+\frac{1}{4}) = 0$ and $g(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(2-\frac{5}{4}-1+\frac{1}{4}) = 0$ so $(\frac{1}{2}, \frac{1}{2})$ is an eq.

8. To calculate the Jacobian we need to take the partial derivatives of those

f and g functions.

$$\begin{aligned}
\frac{\partial p}{\partial f} &= p + 3q - 5pq - 1 + q^2 + p(1-5q) \\
&= 2p + 3q - 10pq - 1 + q^2 \\
\frac{\partial q}{\partial f} &= 0 + p(3 - 5p + 2q) \\
&= 3p + 2pq - 5p^2 \\
\frac{\partial p}{\partial g} &= 0 + q(4 - 5q) \\
&= 4q - 5pq \\
\frac{\partial q}{\partial g} &= 4p - 5pq - 1 + q^2 + q(2q - 5p) \\
&= 4p - 10pq - 1 + 3q^2
\end{aligned}$$

So our Jacobian is

$$J = \begin{bmatrix} 2p + 3q - 10pq - 1 + q^2 & 3p + 2pq - 5p^2 \\ 4q - 5pq & 4p - 10pq - 1 + 3q^2 \end{bmatrix}$$

$$J\Big|_{0,0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$D = 1$, $T = -2$, $T^2 - 4D = 0$. So we conclude that $(0, 0)$ is a stable equilibrium of unknown type.

$$J\Big|_{0,1} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

$D = 6$, $T = 5$, $T^2 - 4D = 1$. So we conclude that $(0, 1)$ is an unstable node type equilibrium.

$$J\Big|_{1,0} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

$D = 3$, $T = 4$, $T^2 - 4D = 4$. So we conclude that $(1, 0)$ is an unstable node type equilibrium.

$$J\Big|_{\frac{1}{2},\frac{1}{2}} = \begin{bmatrix} \frac{-3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{-3}{4} \end{bmatrix}$$

$D = 0$, So we can't say anything about $(\frac{1}{2}, \frac{1}{2})$, using the Jacobian.

9. I'll draw a picture for you if you come to the help sessions. To fussy to make a nice picture right now.

10. What is the best response to $p = q = \frac{1}{3}$?

$$\begin{aligned}\Pi((p,q),(\frac{1}{3},\frac{1}{3})) &= p(\frac{1}{3}+1) + q(\frac{4}{3}) + (1-p-q)(\frac{2}{3}+\frac{2}{3}+\frac{1}{3}) \\ &= \frac{4}{3}p + \frac{4}{3}q + (1-p-q)(\frac{5}{3}) \\ &= \frac{5}{3} - \frac{1}{3}p - \frac{1}{3}q\end{aligned}$$

So clearly the best response is when $p = q = 0$

extra workspace

more workspace

Classifying Equilibria with a Jacobian

Suppose you have a system of differential equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f(p,q)$$
$$\frac{\mathrm{d}q}{\mathrm{d}t} = g(p,q)$$

Then the Jacobian of this system is defined to be

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then let $D = (ad - bc)$ and $T = (a + d)$. If the Jacobian is evaluated at an equilibrium point of the system then using D and T we can classify the type of equilibrium as follows.

1. If $D = 0$ can't say anything about the equilibrium from the Jacobian.
2. If $D < 0$ the equilibrium is a saddle point, and hence unstable. No need to compute T.
3. If $D > 0$ and $T < 0$ the equilibrium is asymptotically stable.
4. If $D > 0$ and $T > 0$ the equilibrium is asymptotically unstable.
5. If $D > 0$ and $T = 0$ the equilibrium point is centre, which is stable but not asymptotically stable. (i.e. not the kind of stable that corresponds to an ESS)
6. If $D > 0$ and $T^2 - 4D < 0$ the equilibrium point is a focus
7. If $D > 0$ and $T^2 - 4D \geq 0$ the equilibrium is a node.

Maximizing a function of two variables

Let $f(p,q)$ be a function of two variables. It's critical points are those which satisfy both of the following equations.

$$\frac{\partial f}{\partial p} = 0$$
$$\frac{\partial f}{\partial q} = 0$$

To check if a critical point is a local maximum, local minimum, or a saddle point you need to look at the matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial p^2} & \frac{\partial^2 f}{\partial p \partial q} \\ \frac{\partial^2 f}{\partial q \partial p} & \frac{\partial^2 f}{\partial q^2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then let $D = ad - bc$ if $D < 0$ the critical point is a saddle point, if $D > 0$ and $\frac{\partial^2 f}{\partial p^2} < 0$ then the critical point is a maximum, if $D > 0$ and $\frac{\partial^2 f}{\partial p^2} > 0$ then the critical point is a minimum.

Binomial distribution

Suppose a coin has a p probability of coming up heads and it is flipped n times. Then the probability of exactly i flips coming up heads is given by $\binom{n}{i}p^i(1-p)^i$. If your calculator doesn't do this for you automatically $\binom{n}{i} = \frac{n!}{(n-i)!\cdot i!}$. This might be helpful in problem one.